Engineering Notes

Assumptions Underlying the "Thin-Skin" Heat-Transfer Approximation

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Nomenclature

b = thickness of skin
B = Biot modulus, $B = h_{ref}b/k$
C = specific heat of skin
h = isothermal heat-transfer coefficient
h_{ref} = isothermal heat-transfer coefficient at reference point
$\bar{h}$ = h/h_{ref}
k = thermal conductivity of skin
K = kernel in heat-transfer integral, see Eq. (10)
l = length of skin in x direction
q = aerodynamic heat-transfer rate
R_0 = outside radius of cylindrical or spherical shell
R_i = inner radius of cylindrical or spherical shell
t = time
t_s = short time lag due to transverse conduction
t_l = longitudinal conduction time lag
T = temperature
T_{aw} = local adiabatic wall (recovery) temperature
x = coordinate along skin in flow direction
y = coordinate normal to skin; $y = 0$ is heated side, $y = -b$ is interior side
ξ = x/l
ρ = mass density of skin
τ = nondimensional time

THE temperature of an aerodynamically heated skin of a flight vehicle is commonly estimated by means of the "thin-skin" approximation. This approximation results from equating the heat stored per unit area in a skin of thickness b with the heat transferred to the skin. The former is $\rho Cb(\partial T/\partial t)$. The latter is based upon the isothermal heat-transfer rate $h(T_{aw} - T)$. In these two terms T is the surface temperature, T_{aw} is the adiabatic wall temperature, h is the aerodynamic heat-transfer coefficient, ρ is the skin density, C its specific heat, and t is time. The material properties are usually assumed to be constant. The resulting differential equation for skin temperature is

$$\rho Cb(\partial T/\partial t) + hT = hT_{aw} \tag{1}$$

In deriving Eq. (1), the temperature is assumed to be uniform across the skin, i.e., all of the heat conduction is across the skin and none is along the skin. The purpose of this discussion is to present additional assumptions that are implicit in the "thin-skin" approximation.

First, the assumption that the temperature is uniform across the thickness of the skin implies that the characteristics of the trajectory do not change rapidly when compared with the transverse conduction time scale[1, 2] denoted t_s

$$t_s = (3.5b^2/\pi^2)\rho(C/k) \tag{2}$$

where k is the thermal conductivity of the skin. t_s is, approximately, the time lag from front to rear skin surfaces. As an example, the value of $(\rho C/k)$ for nickel at room temperature is approximately 1.85 hr/ft^2, and so $t_s \sim 6.54\ b^2$ hr. If the skin is 0.01 ft thick, $t_s \simeq 6.54 \times 10^{-4}$ hr or $\simeq$2.35 sec. For this example, if T_{aw} is changing markedly in 2 sec, then the approximation is poor because of short time heating effects.

The next implicit assumption is that the effects of longitudinal conduction are negligible. Conduction along the skin can be estimated by the following procedure. Assume that the temperature can be represented by the first three terms of a Taylor's expansion about the heated side ($y = 0$), so that

$$T(x, y, t) = T(x, 0, t) + (\partial/\partial y)[T(x, y, t)]\Big|_{y=0} \times y + (\partial^2/\partial y^2)[T(x, y, t)]\Big|_{y=0} (y^2/2!) + \ldots O(y^3) \tag{3}$$

The second derivative of the temperature in Eq. (3) is computed from the boundary condition at $y = -b$, i.e., there is no heat transfer at the back ($y = -b$). The value of the first derivative of the temperature in Eq. (3) is determined from the aerodynamic heat-transfer formula. Equation (3) may be written, after these calculations are made, as

$$T(x, y, t) \simeq T(x, 0, t) + (h/k) \times [T_{aw}(x, t) - T(x, 0, t)][y + (y^2/2b)] \tag{4}$$

The surface temperature is defined by the solution of a differential equation that results from substituting Eq. (4) into the heat-conduction equation

$$k[(\partial^2 T/\partial x^2) + (\partial^2 T/\partial y^2)] = \rho C(\partial T/\partial t) \tag{5}$$

and integrating from $y = -b$ to $y = 0$. Thus one obtains the differential equation

$$\rho Cb\frac{\partial T_0}{\partial t} + h(T_0 - T_{aw}) = kb\frac{\partial^2 T_0}{\partial x^2} - \frac{b^2}{3}\left[k\frac{\partial^2}{\partial x^2} - \rho C\frac{\partial}{\partial t}\right]\left[\frac{h}{k}(T_{aw} - T_0)\right] \tag{6}$$

whose solution defines T_0, where $T_0 = T(x, 0, t)$ [(note that h can be $h(x, t)$]. Clearly, if the right-hand side of Eq. (6) is equal to zero, then Eq. (6) reduces to Eq. (1). To estimate the effects of conduction, define $\xi = x/l$, where l is the panel length in the x direction, and

$$d\tau = \frac{h_{ref}dt}{\rho Cb} \qquad \bar{h} = \frac{h}{h_{ref}} = \frac{h(\xi, t)}{h_{ref}(t)}$$

h_{ref} is a reference value of $\bar{h}$; hence $\bar{h}$ is a function of ξ only and is usually of order unity.†

In terms of these normalized variables, Eq. (6) is

$$\frac{\partial T_0}{\partial \tau} + \bar{h}(T_0 - T_{aw}) - \frac{kb}{h_{ref}l^2}\frac{\partial^2 T_0}{\partial \xi^2} = \left\{-\frac{1}{3\bar{h}}\frac{b^2}{l^2}\frac{\partial^2}{\partial \xi^2} + \frac{h_{ref}b}{3k}\frac{\partial}{\partial \tau}\right\}\{\bar{h}(T_{aw} - T_0)\} \tag{7}$$

† This statement is meant to exclude the case of a mathematically sharp leading edge where h is of the order of $x^{-1/2}$. As $x \to 0$, the surface temperature approaches the adiabatic temperature and longitudinal conduction cannot be neglected.

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The first term on the right-hand side of Eq. (7) represents the heat gain due to internal conduction along the direction of the air flow, and the second term represents a loss or lag in heat resulting from the temperature difference across the thickness. Equation (7) may be rewritten as

$$\frac{\partial}{\partial \tau}\left(1 + \bar{h}\,\frac{B}{3}\right)T_0 + \bar{h}T_0 - \frac{1}{B}\left(\frac{b}{l}\right)^2\frac{\partial^2 T_0}{\partial \xi^2} - \frac{1}{3\bar{h}}\frac{b^2}{l^2}\frac{\partial^2}{\partial \xi^2}(\bar{h}T_0) = \bar{h}T_{aw} + \left[\frac{B}{3}\frac{\partial}{\partial \tau} - \frac{1}{3\bar{h}}\frac{b^2}{l^2}\frac{\partial^2}{\partial \xi^2}(\bar{h}T_{aw})\right] \quad (8)$$

$B = h_{ref}b/k$ is called the Biot modulus and represents the relative ability of the heat to be transferred aerodynamically as compared with its ability to be conducted across the skin. Note that the coefficient of T_0 in the time derivative $(1 + \bar{h}B/3)$ approaches unity if B is small. Hence another condition for the validity of the thin-skin approximation is that $B \ll 1$. If this is the case, and if $\bar{h}$ is well behaved, then

$$(1/3\bar{h})(\partial^2/\partial\xi^2)(\bar{h}T_0) \ll (1/B)(\partial^2 T_0/\partial\xi^2)$$

A further requirement for the use of the thin-skin approximation is that

$$b^2/Bl^2 \ll 1 \quad \text{or} \quad b^2/l^2 \ll B$$

Under these two circumstances, Eq. (8) becomes identical with Eq. (1), the added terms in the forcing function being negligibly small if the derivatives are regular. Note that, by assuming that the term $b^2/Bl^2(\partial^2 T_0/\partial\xi^2)$ is negligible, the highest derivative term in ξ is being neglected. This implies that a boundary-layer-like phenomona may be encountered at the ends ($\xi = 0$ and $\xi = 1.0$) of the panel. The thickness (say δ) of this layer is of the order $\delta/l = [b/l(B)^{1/2}]$. If the duration of the exposure to the hot gas stream is long enough, this boundary layer will include an appreciable part of the panel length. This long-time duration is approximately equal to that given in Eq. (2) if b is replaced by $l/2$, that is,

$$t_l \simeq (3.5/\gamma^2)(l/2)^2(\rho C/k) \quad (9)$$

The 6-in. panel discussed previously would have an upper time constant of $(0.25/0.01)^2 \times 225 = 1406$ sec or 23 min. Thus the characteristics of the trajectory should not change rapidly in 2.35 sec and the entire flight should be completed in less than 23 min.

Another implicit assumption in the use of the thin-skin approximation is contained in the use of the isothermal aerodynamic heat-transfer coefficient. The restrictions resulting from this assumption may be removed by using an integral relation between the aerodynamic heat transfer and the temperature that has the form

$$q = \int_0^x K(x, x_1)d[T_{aw}(x_1) - T_0(x_1)] \quad (10)$$

for either laminar or turbulent flow.[3] Equation (10) may be used to show that the isothermal approximation is valid whenever $T \ll T_{aw}$. When $T \sim T_{aw}$, the integral relation (or its equivalent) must be used if accurate results are required. This is shown by the results of Bryson and Edwards.[4]

Finally, the derivation omits the effects of radiant heat transfer. This may be accounted for by including the radiant heat transfer in the heat balance [Eq. (1)]. Naturally, if the surface is not very wide as compared to its length, a lateral time constant should also be computed.

Equation (8) indicates that the thin-skin approximation has further utility when conduction effects are negligible but B is not too small. In that case, define a time constant of H

$$H = \frac{\bar{h}}{1 + \bar{h}(B/3)}$$

to be introduced in place of $\bar{h}$ in Eq. (8). This time constant can be determined from experimental data. The H so determined then includes many internal effects previously neglected. H can be applied to the problem of calculating thermal response for other trajectories if h is computed for the new trajectory. The modified equation takes the form

$$\frac{\partial}{\partial \tau}\left[1 + \bar{h}\,\frac{B}{3}\right]T_0 + H\left(1 + \bar{h}\,\frac{B}{3}\right)T_0 = \bar{h}T_{aw} + \frac{B}{3}\frac{\partial}{\partial \tau}(\bar{h}T_{aw}) \quad (11)$$

The solution to Eq. (11) may be written in integral form as

$$T_0 = \frac{1}{1 + \bar{h}\,(B/3)}\int_0^\tau \left\{\exp - \left[\int_{\tau_0}^\tau H(\tau_1)d\tau_1\right]\right\} \times \left\{\bar{h}T_{aw} + \frac{B}{3}\frac{\partial}{\partial \tau_0}(hT_{aw})\right\} d\tau_0 \quad (12)$$

where τ_1, τ_0 are dummy variables of integration. In many cases

$$\int_{\tau_0}^\tau H(\tau_1)d\tau_1 \simeq H(\tau - \tau_0)$$

In terms of physical values,

$$H(\tau - \tau_0) = \frac{\bar{h}}{1 + \frac{1}{3}\bar{h}B}\int_{t_0}^t \frac{h_{ref}}{\rho C b}\,d\theta \quad (13)$$

where θ is a dummy variable of integration.

If this approximation is applied to cylindrical or hemispherical shells instead of a flat panel, the factor $\frac{1}{3}$ that multiplies $\bar{h}B$ should be replaced by $\frac{1}{2}$. This is a valid substitution whenever $(R_0 - R_i/R_i)^2 \ll 1$. Here R_0 is the outside radius, R_i is the inside radius, and now $b = R_0 - R_i$.

In summary, the thin-skin approximation will generally be valid whenever the flight time scale falls between t_s and t_l, which are defined by Eqs. (2) and (9). Further, the Biot modulus B should be small as compared to unity but large as compared with the square of the thickness ratio of the panel. Last, the wall temperature should be a small fraction of the adiabatic wall temperature.

The thin-skin approximation is simple to use. Frequently it is applied either directly or in modified form in spite of the systematic errors introduced because it does provide a quick estimate of the expected temperature.

References

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